Reduced-Order Models of Finite Element Approximations of Electromagnetic Devices Exhibiting Statistical Variability

Prasad Sumant, Hong Wu, Andreas Cangellaris, *Fellow, IEEE*, and Narayana Aluru, *Member, IEEE*

***Abstract*—A methodology is proposed for the development of reduced-order models of finite element approximations of electromagnetic devices exhibiting uncertainty or statistical variability in their input parameters. In this approach, the reduced order system matrices are represented in terms of their orthogonal polynomial chaos expansions on the probability space defined by the input random variables. The coefficients of these polynomials, which are matrices, are obtained through the repeated, deterministic model order reduction of finite element models generated for specific values of the input random parameters. These values are chosen efficiently in a multi-dimensional grid using a Smolyak algorithm. The generated stochastic reduced order model is represented in the form of an augmented system that lends itself to the direct generation of the desired statistics of the device response. The accuracy and efficiency of the proposed method is demonstrated through its application to the reduced-order finite element modeling of a terminated coaxial cable and a circular wire loop antenna.**

***Index Terms*—Finite element, Krylov methods, Model order reduction, polynomial chaos, random input, stochastic.**

I. Introduction

FOLLOWING a growing trend in a variety of engineering fields, model order reduction (MOR) has become one of the methods of choice for tackling the complexity of numerical electromagnetic field modeling and simulation inside domains of large geometric and material complexity. The basic idea of MOR is to replace the original discrete system resulting from the discretization of Maxwell's equations over the domain of interest, the dimension of which is too large for an expedient numerical solution to be possible, in terms of one of much smaller dimension, yet capable of approximating the electromagnetic behavior of the original one with sufficient accuracy [1]–[8]. In addition to their application for expediting the calculation of the broadband response of passive electromagnetic systems (a process often referred to as "fast frequency sweep"), MOR has been demonstrated as an efficient technique for use in conjunction with domain decomposition modeling of large-scale electromagnetic structures (e.g., [9]) and for the purposes of sub-domain macro-modeling for the purpose of alleviating the complexity of electromagnetic analysis of multi-scale, passive electromagnetic structures (e.g., [10], [11]).

In their majority, MOR applications for electromagnetic field modeling reported in the literature consider deterministic structures of known material properties and geometry. For those cases where some of the input parameters are allowed to vary in a deterministic fashion, schemes for parameterized model order reduction have been proposed and demonstrated (e.g., [12], [13]). However, it is often the case that geometric and/or material variability of either random or difficult to predict nature may be present in a structure under investigation. For example, surface roughness is a source of modeling uncertainty pertinent to a variety of electromagnetic applications, with notable examples the prediction of attenuation in metallic electromagnetic waveguides and the analysis of electromagnetic scattering by rough surfaces. Another source of modeling uncertainty is the micro/nanomanufacturing process-induced geometric and material variability in the very-fine-scale features encountered in integrated microwave/millimeter-wave and optical devices and circuits, as well as in the context of synthesized metamaterials. The predictive electromagnetic analysis of such structures, accounting for the impact of material and geometric uncertainty, is becoming a growing need in the computer-aided exploration, design, and prototyping of such devices and synthesized materials.

Of particular interest to this work is the case where one or several portions (or domains) of a complex structure exhibit such statistical variability or uncertainty in the parameters that define their geometric and material composition. In the following, such domains will be referred to as "stochastic" domains. For such structures, the development of stochastic, reduced-order models of the stochastic domains offers an interesting option for facilitating and expediting system-level electromagnetic analysis in the presence of uncertainty. The demonstration of a methodology for the development of stochastic reduced order models of finite element approximations of passive electromagnetic structures is the objective of this paper.

More specifically, our objective is to put forward a method that treats MOR as a black-box deterministic solver; thus,

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P. Sumant was with the Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign, IL 61820 USA. He is now with ExxonMobil Upstream Research Company, Houston, TX 77098 USA (e-mail: ps.sumant@gmail.com).

H. Wu is with Extreme DA Corporation, Santa Clara, CA 95054 USA.

A. Cangellaris and N. Aluru are with University of Illinois at Urbana-Champaign, IL 61820 USA.

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the method is independent of the specific MOR algorithm employed. Our methodology takes advantage of recent advances in the ideas of generalized polynomial chaos and stochastic collocation for the polynomial interpolation of the random variables involved and the establishment of a framework for the mathematical representation of the stochastic reduced-order model. Among the numerous contributions to the rapidly growing literature on the generalized polynomial chaos methodology for stochastic computation, the review article by Xiu [14] is worth mentioning because of its comprehensive bibliography on the subject.

The rest of the paper is organized as follows. The mathematical framework is developed in Section II. Following a brief review of the deterministic MOR process and the use of polynomial chaos as a means of representing uncertainty, the proposed methodology for stochastic MOR is presented for the specific case of a finite element approximation of Maxwell's equations in a linear, isotropic domain. The validity and accuracy of the proposed methodology are discussed in Section III in the context of its application to the reduced-order modeling of terminated coaxial cable and a circular loop antenna. The paper concludes with a summary of the contribution and a discussion of topics of on-going research.

II. Mathematical Formulation

A. Deterministic Model Order Reduction

In order to keep the mathematical framework simple, we restrict ourselves to the finite element approximation of a linear, electromagnetic boundary value problem inside a domain with isotropic, non-dispersive media. However, it should be noted that the proposed methodology is also applicable when the media is anisotropic. As far as the assumption of non-dispersive media, this is driven by our intention to demonstrate in a simple manner the fact that the proposed methodology is applicable to the reduced-order modeling in both the frequency and the time domain. For the case of dispersive media, the model order reduction framework presented in [9], [15] must be employed in place of the simpler, reduced-order modeling algorithm presented here.

Following [2], we develop a finite element approximation of the vector Helmholtz equation for the electric field using edge elements. It is assumed that the structure under modeling interacts with its exterior through N_p ports. Hence, the Laplace-domain representation of the finite element model of the N_p-port system is described by the following system of equations [2]

$$\begin{aligned}(Y + sZ + s^2Pe)x &= sBI \\ y &= L^H x\end{aligned} \tag{1}$$

where N is the number of degrees of freedom in the finite element approximation, the vector x contains the coefficients in the finite element approximation of the electric field, the matrices Y, Z, Pe are in $R^{N\times N}$, the matrices B, L are in $C^{N\times N_p}$, the vectors I, y are in C^{N_p} and s is the complex angular frequency. The vector I indicates unit excitation at each port. The matrix B is dependent on the port characteristics and is used to map the unit excitation to the state space variables. Similarly, the matrix L is used to sample the calculated electric field to generate the desirable output quantity. The generalized multi-port impedance matrix (GIM) is given by [2],

$$Z_G(s) = sL^H(Y + sZ + s^2Pe)^{-1}B \tag{2}$$

Calculation of the GIM at a given frequency requires the inversion of a finite element matrix that is of large dimension, typically in the order of several tens or even hundreds of thousands. This is where MOR offers substantial reductions in cost. A reduced-order model attempts to approximate the original system in terms of a subset of the vectors in the eigenspace of the full finite element system. More specifically, the eigenvectors of interest are the ones that are the most influential in the calculation of the response of the structure as seen by the ports involved in the definition of Z_G.

As mentioned in the introduction and as it will be demonstrated later in this section, the proposed methodology for stochastic MOR is such that it is compatible with anyone of the popular approaches used in practice for deterministic MOR. For our purposes, the Krylov subspace algorithm described in [2] will be used. In this approach, a transformation matrix F in $C^{N\times q}$ is constructed, where $q \ll N$. The reduced order system, which is of dimension q, can then be computed as,

$$\hat{Z}_G(s) = s\hat{L}^H(\hat{Y} + s\hat{Z} + s^2\hat{P}e)^{-1}\hat{B} \tag{3}$$

where,

$$\begin{aligned}&\hat{Y} = F^HYF, \quad \hat{Z} = F^HZF \quad \hat{P}e = F^HPeF \\ &\hat{B} = F^HB \quad \hat{L} = F^HL.\end{aligned} \tag{4}$$

Clearly, the matrices $\hat{Y}, \hat{Z}, \hat{P}e$ are in $R^{q\times q}$.

B. Polynomial Chaos Expansion

Next, we consider the way statistical variability is introduced in the finite element model. Toward this, let D denote the physical domain of interest with $\vec{r}$ denoting the position vector in D. To keep the presentation simple, and without loss of generality, we assume that the uncertainty in the definition of the electromagnetic boundary problem of interest can be described in terms of two continuous, orthogonal random variables, $\xi_1(\theta)$, $\xi_2(\theta)$, where θ represents an event in our probability space. For random variables that are correlated, they can always be transformed into orthogonal random variables using procedures such as Gram-Schmidt orthogonalization. Let Γ_i denote the support of ξ_i, $i = 1, 2$. There are three kinds of support, Γ_i, namely, the bounded support in $(-1, 1)$, the half-space support $(0, +\infty)$, and the whole space support $(-\infty, +\infty)$. Let $\rho_i(\xi_i)$, be the probability density function (PDF) of the random variable $\xi_i(\theta)$, $i = 1, 2$. Then, the joint probability density of (ξ_1, ξ_2) is $\rho(\xi_1, \xi_2) = \rho_1(\xi_1)\rho_2(\xi_2)$, with support $\Gamma = \Gamma_1 \times \Gamma_2$. In this context, the electromagnetic field response of interest for a given excitation is the random process $\vec{E}(\vec{r}, \theta)$ in $D \otimes \Gamma$.

Before we proceed with the discussion of polynomial interpolation in the two-dimensional random space Γ, it is worth noting that the description of the uncertainty or statistical variability in terms of a finite set of random variables is one of the most important steps in the pursuit of a meaningful solution to the boundary

value problem of interest. While statistical variability of constitutive material parameters is, in most cases, easily quantified in terms of random variable, variability or uncertainty in geometrical features and boundary conditions often pose challenges in their modeling in terms of random variables. An overview of approaches in use today for tackling such challenges is provided in [14].

The basic idea of polynomial chaos is to approximate a random function in Γ in terms of orthogonal polynomial expansions. Let $\psi_m(\xi_i)$, $m = 0, 1, 2, \ldots, d_i$, be a set of orthogonal polynomials of maximum degree d_i, for interpolation in ξ_i. The pertinent orthogonality condition is

$$\langle \psi_m(\xi_i), \psi_n(\xi_i) \rangle = \delta_{mn} \langle \psi_m^2(\xi_i) \rangle \tag{5}$$

$\langle \cdot, \cdot \rangle$ denotes the inner product given by,

$$\langle \psi_m(\xi_i), \psi_n(\xi_i) \rangle = \int_{\Gamma_i} \rho_i(\xi_i) \psi_m(\xi_i) \psi_n(\xi_i) d\xi_i. \tag{6}$$

It is noted that the PDF ρ_i serves as the integration weight in the orthogonality relation and defines the class of orthogonal polynomials. For example, for a uniformly distributed random variable in $(-1, 1)$, the PDF is a constant and the orthogonality relation defines Legendre polynomials. For the case of a Gaussian distribution, the orthogonality relation defines Hermite polynomials. The examples presented in the next section make use of uniform and Gaussian random variables; thus, Legendre and Hermite polynomials are the two classes of polynomials that will be used in our polynomial chaos expansions.

The extension to polynomial interpolation in the two-dimensional space Γ is straightforward. Thus, we define the polynomial space $\Psi_m(\xi_1, \xi_2)$, $m \leq P$, with elements products of polynomials in the two random variables

$$\Psi_m(\xi_1, \xi_2) = \psi_{m_1}(\xi_1) \psi_{m_2}(\xi_2), m_1 + m_2 \leq P \tag{7}$$

where, $0 \leq m \leq M - 1$, with M being the number of expansion functions in the polynomial space thus defined. M is given by

$$M = \begin{pmatrix} N_d + P \\ N_d \end{pmatrix} \tag{8}$$

where, N_d denotes the dimension of the polynomial space $(N_d = 2)$. In view of the orthogonality of the polynomials, it is

$$\langle \Psi_m(\xi_1, \xi_2), \Psi_n(\xi_1, \xi_2) \rangle = \delta_{mn} \langle \Psi_m^2(\xi_1, \xi_2) \rangle \tag{9}$$

$\langle \cdot, \cdot \rangle$ denotes the inner product given by,

$$\langle \Psi_m, \Psi_n \rangle = \int_{\Gamma} \Psi_m(\xi_1, \xi_2) \Psi_n(\xi_1, \xi_2) \rho(\xi_1, \xi_2) d\xi_1 d\xi_2. \tag{10}$$

Equipped with the polynomial expansion functions in the random space Γ, the electric field unknown at a position $\vec{r} \in D$ is approximated as follows

$$u(\vec{r}, s, \xi_1, \xi_2) = \sum_{m=0}^{M-1} \widehat{a_m}(\vec{r}, s) \Psi_m(\xi_1, \xi_2). \tag{11}$$

Similar expansions are used for the approximation of any quantity influenced by the uncertainty in the computational domain. For example, for $P = 1$ and for the case of Legendre polynomials, linear expansions are used to approximate the stochastic matrices in (1). For example, the matrix $\tilde{Y}$, is approximated by

$$Y \approx \tilde{Y} = Y_0 + Y_1 \xi_1 + Y_2 \xi_2. \tag{12}$$

It is noted that the coefficient matrices in the above expression are deterministic. Once they are found, the stochastic matrix is completely defined. The process we follow for the calculation of these coefficient matrices is discussed later in this section.

C. Stochastic Model Order Reduction

Let the stochastic reduced order system be represented by,

$$\begin{aligned} (\tilde{\hat{Y}} + s\tilde{\hat{Z}} + s^2 \tilde{\hat{P}}e)\tilde{\hat{x}} &= s\tilde{\hat{B}} I \\ \tilde{\hat{y}} &= \tilde{\hat{L}}^H \tilde{\hat{x}} \end{aligned} \tag{13}$$

with

$$\begin{aligned} &\tilde{\hat{Y}} = \tilde{F}^H \tilde{Y} \tilde{F}, \quad \tilde{\hat{Z}} = \tilde{F}^H \tilde{Z} \tilde{F}, \quad \tilde{\hat{P}}e = \tilde{F}^T \tilde{P} e \tilde{F} \\ &\tilde{\hat{L}} = \tilde{F}^H \tilde{L}, \quad \tilde{\hat{B}} = \tilde{F}^H \tilde{B}. \end{aligned} \tag{14}$$

Clearly, the development of the stochastic reduced-order model requires the construction of the stochastic projection matrix $\tilde{F}$. However, before we delve into the process used for its construction, let us assume that the matrix has been constructed and is available in terms of a polynomial chaos expansion over the probability space Γ. In a similar manner, assume that polynomial chaos expansions are available for the stochastic matrices of the original, full finite element system. For example, for the case of a linear approximation using Legendre polynomials, these expansions are of the form

$$\begin{aligned} \tilde{Y} &= Y_0 + Y_1 \xi_1 + Y_2 \xi_2 \\ \tilde{Z} &= Z_0 + Z_1 \xi_1 + Z_2 \xi_2 \\ \tilde{P}e &= P_0 + P_1 \xi_1 + P_2 \xi_2 \\ \tilde{F} &= F_0 + F_1 \xi_1 + F_2 \xi_2. \end{aligned} \tag{15}$$

For now, we assume that all the coefficient matrices in the above expansions are known. Using these in (14), the matrices of the stochastic reduced-order model are readily obtained. For example, the $\tilde{\hat{Y}}$ matrix is obtained as

$$\begin{aligned} \tilde{\hat{Y}} = &(F_0 + F_1 \xi_1 + F_2 \xi_2)^H \\ &\times (Y_0 + Y_1 \xi_1 + Y_2 \xi_2)(F_0 + F_1 \xi_1 + F_2 \xi_2) \end{aligned}$$

with similar expansions for the remaining matrices. The R.H.S. in the above expression can be expanded and terms arranged such that they can be cast in the form of orthogonal polynomial expansion. The resulting expressions may be truncated to the desirable degree of polynomial chaos. For instance,

$$\begin{aligned} \tilde{\hat{Y}} = &\left(F_0^H Y_0 F_0\right) \\ &+ \left(F_0^H Y_0 F_1 + F_0^H Y_1 F_0 + F_1^H Y_0 F_0\right) \xi_1 \\ &+ \left(F_0^H Y_0 F_2 + F_0^H Y_2 F_0 + F_2^H Y_0 F_0\right) \xi_2 \\ &+ \cdots. \end{aligned}$$

By utilizing a similar linear polynomial chaos for the expansion of the reduced-order model matrices and vectors on the two-

dimensional probability space Γ as that of the full system, the reduced stochastic matrix $\tilde{\hat{Y}}$ is approximated as follows:

$$\tilde{\hat{Y}} = \hat{Y}_0 + \hat{Y}_1\xi_1 + \hat{Y}_2\xi_2 \tag{16}$$

where the coefficient matrices are given by the following expressions:

$$\begin{aligned}
&\hat{Y}_0 = F_0^H Y_0 F_0\\
&\hat{Y}_1 = \left(F_0^H Y_0 F_1 + F_0^H Y_1 F_0 + F_1^H Y_0 F_0\right)\\
&\hat{Y}_2 = \left(F_0^H Y_0 F_2 + F_0^H Y_2 F_0 + F_2^H Y_0 F_0\right)
\end{aligned} \tag{17}$$

$$\begin{aligned}
&[(\hat{Y}_0 + \hat{Y}_1\xi_1 + \hat{Y}_2\xi_2) + s(\hat{Z}_0 + \hat{Z}_1\xi_1 + \hat{Z}_2\xi_2)\\
&\quad + s^2(\hat{P}_0 + \hat{P}_1\xi_1 + \hat{P}_2\xi_2)](\hat{x}_0 + \hat{x}_1\xi_1 + \hat{x}_2\xi_2)\\
&\quad\quad = s(\hat{B}_0 + \hat{B}_1\xi_1 + \hat{B}_2\xi_2)I
\end{aligned}$$

$$\tilde{\hat{y}} = (\hat{L}_0 + \hat{L}_1\xi_1 + \hat{L}_2\xi_2)^H(\hat{x}_0 + \hat{x}_1\xi_1 + \hat{x}_2\xi_2) \tag{18}$$

$$\begin{aligned}
&\begin{bmatrix} \hat{Y}_0 & \hat{Y}_1/3 & \hat{Y}_2/3\\ \hat{Y}_1 & \hat{Y}_0 & 0\\ \hat{Y}_2 & 0 & \hat{Y}_0 \end{bmatrix}\begin{bmatrix} \hat{x}_0\\ \hat{x}_1\\ \hat{x}_2 \end{bmatrix}\\
&\quad + s\begin{bmatrix} \hat{Z}_0 & \hat{Z}_1/3 & \hat{Z}_2/3\\ \hat{Z}_1 & \hat{Z}_0 & 0\\ \hat{Z}_2 & 0 & \hat{Z}_0 \end{bmatrix}\begin{bmatrix} \hat{x}_0\\ \hat{x}_1\\ \hat{x}_2 \end{bmatrix}\\
&\quad + s^2\begin{bmatrix} \hat{P}_0 & \hat{P}_1/3 & \hat{P}_2/3\\ \hat{P}_1 & \hat{P}_0 & 0\\ \hat{P}_2 & 0 & \hat{P}_0 \end{bmatrix}\begin{bmatrix} \hat{x}_0\\ \hat{x}_1\\ \hat{x}_2 \end{bmatrix} = s\begin{bmatrix} \hat{B}_0\\ \hat{B}_1\\ \hat{B}_2 \end{bmatrix} I
\end{aligned}$$

$$\begin{bmatrix} \hat{y}_0\\ \hat{y}_1\\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} \hat{L}_0^H & 0 & 0\\ \hat{L}_1^H & \hat{L}_0^H & 0\\ \hat{L}_2^H & 0 & \hat{L}_0^H \end{bmatrix}\begin{bmatrix} \hat{x}_0\\ \hat{x}_1\\ \hat{x}_2 \end{bmatrix}. \tag{19}$$

Similar expansions can be written for the rest of the matrices in the reduced-order model. Using a linear polynomial chaos expansion for the output $\tilde{\hat{y}}$, we have

$$\tilde{\hat{y}} = \hat{y}_0 + \hat{y}_1\xi_1 + \hat{y}_2\xi_2.$$

Thus, combining this with (13) and (16) the stochastic reduced-order system assumes the form of (18). By rearranging the terms and making use of the orthogonality of the polynomials, (18) may be cast in terms of the deterministic system in (19). Clearly, the dimension of this system is $3q$. In a compact form, we write

$$\begin{aligned}
Y_{\text{aug}} + sZ_{\text{aug}} + s^2P_{\text{aug}}x_{\text{aug}} &= sB_{\text{aug}}I\\
y_{\text{aug}} &= L_{\text{aug}}^H x_{\text{aug}}
\end{aligned} \tag{20}$$

where the correspondence between the names of all matrices and vectors between the two sets of equations is apparent, while the subscript $_{aug}$ is used to remind us that these matrices and vectors are defined in terms of the coefficients in the polynomial chaos expansions of the stochastic reduced order model. Clearly, the specific form of these matrices depends on both the choice of the polynomials used in the polynomial chaos expansions and the highest polynomial degree kept in the expansions. The ones derived above are for the case of orthogonal Legendre polynomials.

At this point we return to the issue of computing the coefficients in the polynomial chaos expansions of the stochastic projection matrix and full finite element matrices (16). Clearly, the orthogonality relation (9) can be used for this purpose. For example, the coefficient matrix Y_0 in the polynomial chaos expansion of $\tilde{Y}$ is readily computed as follows

$$\begin{aligned}
Y_0 &= \int_\Gamma \tilde{Y}\rho_1(\xi_1)\rho_2(\xi_2)d\xi_1 d\xi_2\\
Y_0 &= \int_\Gamma (Y_0 + Y_1\xi_1 + Y_2\xi_2)\rho_1(\xi_1)\rho_2(\xi_2)d\xi_1 d\xi_2.
\end{aligned} \tag{21}$$

In a similar manner, the remaining coefficients, $Y_i, i = 1, 2$, are obtained through the expression

$$Y_i = \int_\Gamma \tilde{Y}\rho_1(\xi_1)\rho_2(\xi_2)\xi_i d\xi_1 d\xi_2. \tag{22}$$

The efficient calculation of these integrals is discussed next.

D. Smolyak Algorithm for Numerical Integration in Multiple Dimensions

Let $I(f)$ denote one of the integrals in (21), (22). Considering the case of one-dimensional integration first, let us assume, without loss of generality, that the support of each one of the two random variables is the interval $[-1, 1]$. Then $I(f)$ may be computed in terms of an appropriate quadrature rule through a summation of the form

$$I(f) = \sum_{j=1}^{q} f(u^j)w^j. \tag{23}$$

This integration rule involves the calculation of the integrand at q points, u^j, with corresponding weights w^j. Among the various possible choices for the case of a polynomial integration rule, the extrema of the Chebyshev polynomials is a commonly used option. In this case, the sampling points are given by

$$u^j = -\frac{\cos(\pi(j-1))}{q-1}, \quad j = 1\ldots q. \tag{24}$$

In extending the above integration rule to higher dimensions, the key issue is to ensure the accuracy of the numerical integration while at the same time maintain the efficiency of the numerical calculation by tackling the challenge of the exponential growth of the number of sampling points. For the development that follows we adopt the terminology used in [16].

Assume an N_d-dimensional random space and let $I_i^{q_i}$ denote the one-dimensional integration rule in the i dimension, $i = 1, 2, \ldots N_d$, consisting of q_i points on the support interval Γ_i. Thus,

$$I_i^{q_i}[f] = \sum_{j=1}^{q_i} f\left(u_i^j\right)\cdot w_i^j \tag{25}$$

based on nodal sets

$$\Theta_i^1 = \left(u_i^1, \ldots, u_i^{q_i}\right) \subset \Gamma_i. \tag{26}$$

A straightforward approach for extending the above one-dimensional numerical integration to multiple dimensions is through the tensor product

$$\begin{aligned}
I^Q[f] &\equiv \left(I_i^{q_1} \otimes \cdots \otimes I_i^{q_{N_d}}\right)[f]\\
&= \sum_{j_1=1}^{q_1}\cdots\sum_{j_{N_d}=1}^{q_N} f\left(u_1^{j_1}, \ldots, u_{N_d}^{j_{N_d}}\right)
\end{aligned}$$

$$\cdot \left(w_1^{j_1} \otimes \cdots \otimes w_{N_d}^{j_{N_d}} \right). \tag{27}$$

Clearly, the number of terms in the above summation equals the number of nodal points $Q = \prod_{i=1}^{N_d} q_i$ nodal points. If we choose the same number of points in each dimension, i.e., $q_1 = \cdots = q_{N_d} = q$, the total number of points is $Q = q^{N_d}$. The rapid growth of the number of nodes and, consequently, of the computational cost of the integration, when using tensor products is evident. A computationally efficient alternative makes use of sparse grids for multi-dimensional numerical integration. This is discussed next.

Sparse Grids: Sparse grids have been used for solving random differential equations by stochastic collocation approach in multi-dimensional random spaces [17]. For our purposes, we will make use of the Smolyak algorithm ([18]–[20]).

Starting with the one-dimensional integration formula (25), the Smolyak algorithm is given by

$$I^Q(f) \equiv A(J, N_d) = \tag{28}$$

$$\sum_{J-N_d+1 \le |i| \le J} (-1)^{J-|i|} \cdot \binom{N_d - 1}{J - |i|} \cdot (I_{i_1} \otimes \cdots \otimes I_{i_{N_d}}) \tag{29}$$

where $i = (i_1, \ldots, i_{N_d}) \in N_d^{N_d}$. To compute $A(J, N_d)$ we only need to evaluate function on the *sparse grid*

$$\Theta_{N_d} \equiv H(J, N_d) = \bigcup_{J-N_d+1 \le |i| \le J} (\Theta_{i_1}^1 \times \cdots \times \Theta_{i_{N_d}}^1). \tag{30}$$

In this paper, we use the Smolyak formulas based on one-dimensional polynomial integration at the extrema of the Chebyshev polynomials (24). In addition, we define $u_i^1 = 0$ if $q_i = 1$ and choose $q_1 = 1$ and $q_i = 2^{i-1} + 1$ for $i > 1$. This definition makes the one-dimensional nodal sets Θ_i^1 nested, so $H(J, N_d) \subset H(J+1, N_d)$. Let $W_{N_d}^P$ be the space of N_d-variate orthonormal polynomials of total degree at most P. It can be shown that, if we set $J = N_d + P$, then $A(N_d + P, N_d)$ is exact for integration of polynomials in a space larger than $W_{N_d}^P$, and that the total number of nodes, Q, for sufficiently large N_d is given, approximately, by

$$Q \equiv \dim(A(N_d + P, N_d)) \sim \frac{2^P}{P!} N_d^P, P \text{ fixed}, N_d \gg 1. \tag{31}$$

Clearly, the dependence on dimension N_d is much weaker than tensor product rule. Henceforth in this paper, we will refer to k in $A(N_d + k, N_d)$ as the level of the sparse grid integration. As a specific example, Fig. 1 depicts a comparison of two dimensional grids constructed using the tensor product and the Smolyak algorithm for level 4. Clearly, use of the Smolyak algorithm reduces significantly the number of nodes in the numerical calculation of the integral.

With the Smolyak grid defined, the integrals in (21) and (22) for the case of the polynomial chaos expansion of matrix Y of the original system are computed as follows

$$Y_i = \sum_{j=0}^{q} Y^j \rho_1(\xi_1^j) \rho_2(\xi_2^j) \xi_i^j w^j \tag{32}$$

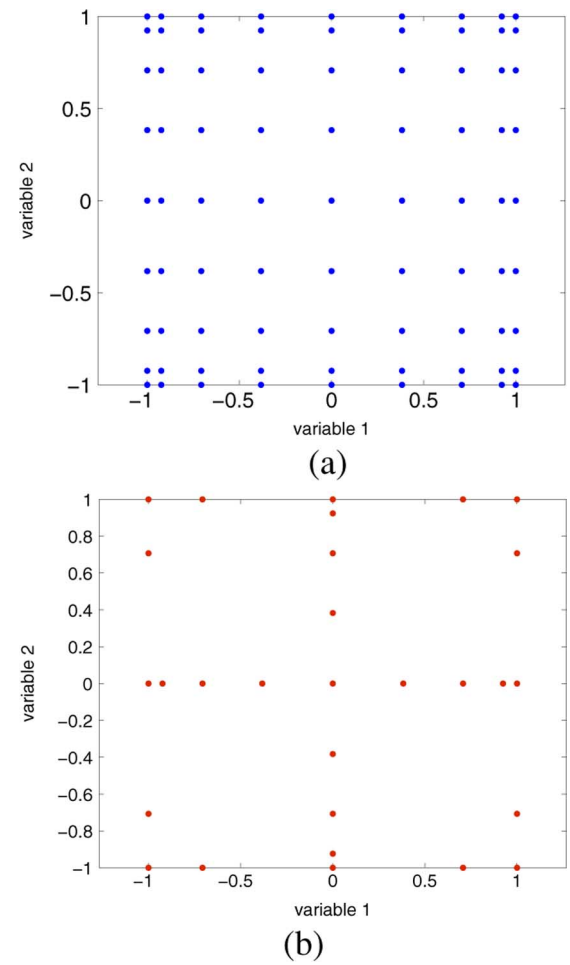

Fig. 1. Grids for two-dimensional integration using the extrema of Chebyshev polynomials. (a) Tensor product grid (total number of points 81). (b) Smolyak grid (total number of points 29).

where Y^j is the deterministic system matrix computed for the specific values of the input random variables at the jth Smolyak grid point, $\xi_1 = \xi_1^j$ and $\xi_2 = \xi_2^j$. The coefficients in the polynomial chaos of all original system matrices are obtained in a similar fashion.

The calculation of the polynomial chaos expansion of the stochastic projection matrices proceeds as follows. For each point (ξ_1^j, ξ_2^j) on the Smolyak grid a deterministic full system is obtained. For this system, using a Krylov subspace method (or any other method of choice) a projection matrix, F^j is obtained. Once these matrices have been constructed, the coefficients in the polynomial chaos expansion of the projection matrix $\tilde{F}$ are computed as follows:

$$F_i = \sum_{j=0}^{q} F^j \rho_1(\xi_1^j) \rho_2(\xi_2^j) \xi_i^j w^j. \tag{33}$$

This completes our presentation of the mathematical framework for stochastic model order reduction. The proposed methodology may be summarized in terms of the series of steps described in Algorithm 1.

Algorithm 1 Algorithm for Stochastic Model Order Reduction

1: Identify statistically varying material and geometric parameters and represent them in terms of independent random variables $\xi = [\xi_1, \xi_2, \ldots, \xi_{N_d}]^T$.
2: Choose an appropriate order and type of polynomial chaos expansion.
3: Generate a Smolyak grid Θ_{N_d} and corresponding weights w_{N_d}. Each point represents a combination of values for different input random variables.
4: For each point on the Smolyak grid, calculate the full finite element system matrices. Perform deterministic MOR for each system to generate the corresponding projection matrix.
5: Using above information, calculate the coefficients in the polynomial chaos expansion of the stochastic full finite element system matrices and the stochastic projection matrix ((32), (33)).
6: Compute the coefficients in the polynomial chaos expansion of the stochastic, reduced matrix (17).
7: Compute the augmented system (20).
8: The augmented system can be used to calculate calculating the mean, standard deviation and other statistics of the system response.

III. Numerical Studies

A. Input Impedance of Terminated Coaxial Cable

The first example considers a structure for which an analytic solution can be obtained, namely, a terminated air-filled coaxial cable. The length of the coaxial cable is $L = 1$ m, and its two electrodes are assumed to be perfectly conducting. The radius of the inner, circular cylindrical electrode is $a = 5$ mm. The inner radius of the outer circular cylindrical electrode is $b = 10$ mm. The cable is terminated at its far end by a $Z_L = 30\ \Omega$ resistor.

First, we would like to investigate the effectiveness of Smolyak sparse grid for performing numerical integration. We make use of the transmission-line theory formula for the input impedance of the terminated cable,

$$Z_{\text{in}}(\omega) = Z_0 \frac{Z_L + jZ_0 \tan(\beta L)}{Z_0 + jZ_L \tan(\beta L)} \tag{34}$$

where, Z_0, is the cable characteristic impedance,

$$Z_0 = \sqrt{\frac{\mu}{\epsilon}} \frac{\ln(b/a)}{2\pi} \tag{35}$$

and $\beta = \omega\sqrt{\mu\epsilon}$ is the propagation constant. We consider two random inputs. One is the permittivity of the insulating medium in the cable, which is modeled as a uniform random variable in the interval [3.6–4.4]. The second is the load resistance, which is modeled as a uniform random variable in [25–35]. With the frequency set at $f = 100$ MHz, using a standard Monte Carlo approach with 102400 random samples, the mean value of the real part of the input impedance is obtained as 16.7386 Ω and its standard deviation as 2.8995 Ω. For sparse grid integration, we consider different levels starting from level 3. Table I summarizes the results. It is clear that the Sparse grid integration converges to the linear expansion solution level 4 onwards. Hereafter in this paper, we make use of level 4 for sparse grid integration.

TABLE I
Performance of Stochastic Collocation

Level	N	Zmean	Zstd
3	13	16.7947	2.83267
4	29	16.7975	2.83574
5	65	16.7975	2.83578
6	145	16.7975	2.83578
7	321	16.7975	2.83578

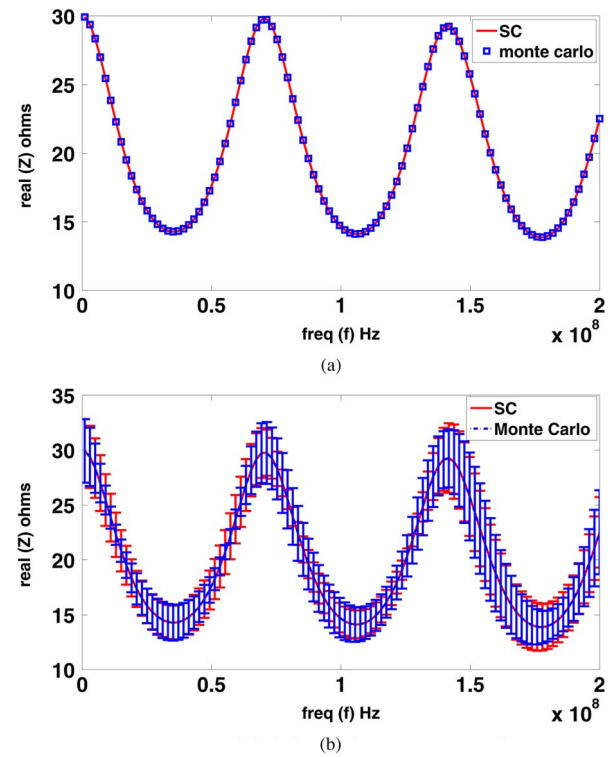

Fig. 2. Real part of input impedance of a coaxial cable. (a) Mean (real part of impedance). (b) Standard deviation (real part of impedance).

A full finite element model was developed for the coaxial cable. The finite element system is of dimension 36840. The generated reduced order model is of order 20. 10201 Monte Carlo simulations are performed to generate the reference solution. For the proposed method, a level 4 Smolyak algorithm is used that involves 29 points in the two-dimensional grid. Hence, only 29 deterministic problem solutions are involved in the sparse grid instead of the 10201 used for the standard Monte Carlo. Shown in Fig. 2 is a comparison of the two approaches in the calculation of the real part of the input impedance of the terminated coaxial cable. Very good agreement is observed.

In practice it is customary to rely on "corner analysis" for assessing the impact of variability of the input parameters on the response. To contrast this approach with that of a stochastic modeling, results for the input impedance, obtained using extreme values of the two stochastic input parameters, are generated. More specifically, three sets of corner values are considered. A minimum values set, $(\epsilon = 3.6, Z_L = 25)$, a

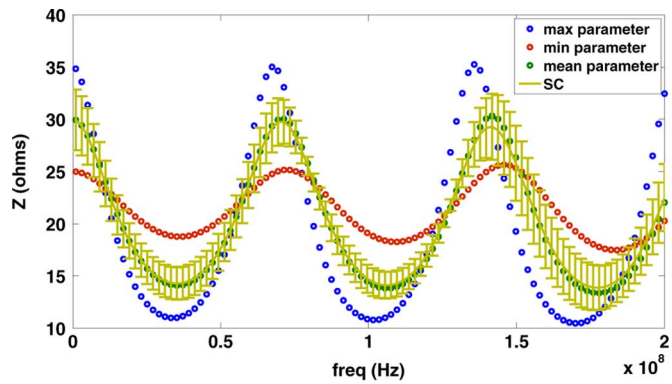

Fig. 3. Comparison of real part of input impedance using extreme values for the random parameters with the statistics obtained using the proposed stochastic MOR approach.

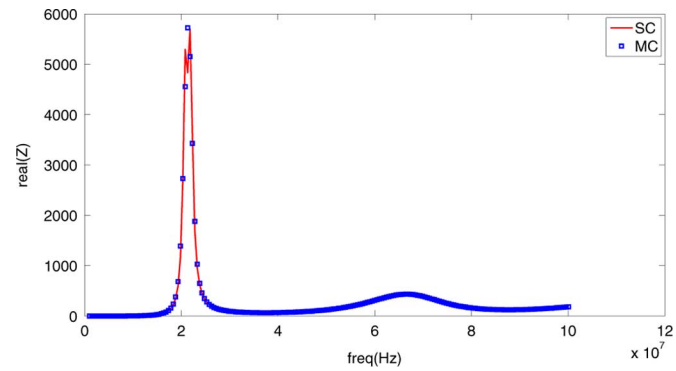

Fig. 4. Comparison of calculated mean value of the real part of the input impedance of the wire loop antenna obtained using the proposed stochastic MOR (SC) and standard Monte Carlo (MC).

mean values set, $(\epsilon = 4.0, Z_L = 30)$, and a maximum values set, $(\epsilon = 4.4, Z_L = 35)$. The calculated real part of the input impedance versus frequency for these three sets is compared in Fig. 3 with the mean response obtained using the proposed algorithm. The error bars indicate one standard deviation. This comparison clearly shows that the statistics of the response generated by the proposed stochastic MOR approach provides a tighter estimate of the system response. It is also worth noting that there is a discrepancy between the response obtained from using mean parameter values from the calculated mean response, a result that is expected for the case where multiple random variables are involved.

B. Circular Wire Loop Antenna

The next numerical study considers the electromagnetic modeling of a circular wire loop antenna in air. The radius of the circular loop is 1 m. The computational domain used for the finite element model is a cube of side 5 m. A first-order absorbing boundary condition was imposed on all six walls of the domain. Further details of the finite element model used can be found in [9]. A reduced-order model of order 30 was generated. We are interested in assessing the impact on the transient response of the antenna of statistical variability in geometric parameters such as the wire radius and the loop radius. The wire radius is modeled as a Gaussian random variable with mean of 0.05 m and standard deviation of 0.01 m. The loop radius is modeled as a Gaussian random variable with mean of 1 m and standard deviation of 0.03 m.

We first establish the accuracy of the proposed method by comparing the mean input impedance with that obtained using Monte Carlo. 9801 Monte Carlo simulations are performed to generate the reference solution. A level 4 in the Smolyak algorithm results in sparse grid sampling with 29 points on the two-dimensional probability space. Shown in Fig. 4 is a comparison between the results produced by our approach and standard Monte Carlo for the real part of the mean input impedance. Of particular interest in this problem is the mean fundamental frequency obtained using the two methods. The fundamental frequency obtained using Monte Carlo is 21.3 MHz as against 21.32 MHz obtained using our proposed algorithm.

Next, we consider the transient response of the loop antenna to a pulse-current source excitation. More specifically, a rectangular current pulse of rise and fall times of 10 ns, pulse width of 60 ns and amplitude of 1 A, is used to excite the wire loop antenna. Plotted in Fig. 5(a) is the mean input voltage obtained using the generated stochastic reduced-order model. Also plotted in the same figure is the input voltage obtained using a deterministic reduced order model employing the mean values for the two input parameters (wire radius and loop radius). We observe that, as time progresses, the deviation between the two responses is appreciable. This highlights the necessity of using a stochastic reduced order model for computing the mean system response when more than one input variables exhibit statistical variability.

Finally, we compare the stochastic system response, plotted in terms of mean voltage V_0 with error-bars corresponding to one standard deviation $\sqrt{V_1^2 + V_2^2}$, with responses from "corner" solutions, obtained using deterministic reduced order models employing minimum and maximum values of the input parameters (Fig. 5(b)). It is clear from Fig. 5(b) that "corner" simulations can be unreliable and that care should be exercised in their use for the development of error bounds in the system response.

IV. Conclusion

In this paper, we have proposed a method for performing model order reduction of passive electromagnetic devices in the presence of uncertainty or statistical variability in geometric or material parameters. The method makes use of generalized polynomial chaos expansions on the probability space defined by the stochastic variables for the finite element system matrices and the projection matrix used in the development of the reduced-order model. The coefficient matrices in these expansions are obtained from a numerical integration using sparse grids of sampling points on the probability space, generated by means of the Smolyak algorithm. At each sparse grid point a deterministic model order reduction operation is carried out to calculate the corresponding transformation matrix. Thus, the computation complexity of the proposed process is dictated by the number of points on the sparse grid. However, thanks to the mutual independence of the model order reduction operations at the sparse grid points, the parallelism of these calculations can be

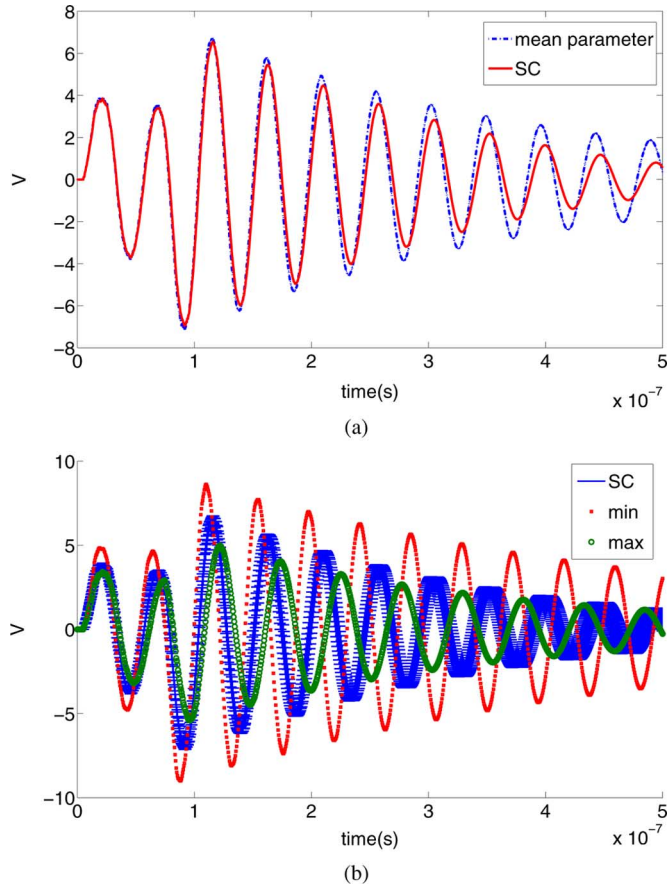

Fig. 5. Loop antenna transient response to a pulse current source excitation. (a) Comparison of mean stochastic response with the response obtained using mean values of the input parameters. (b) Comparison of stochastic response with "corner" responses obtained using extreme values of input parameters.

exploited for containing the computational cost. The stochastic, reduced-order model that is generated is in terms of a polynomial expansion in terms of the uncertain parameters in the model. Thanks to the small dimension of these matrices, the reduced-order system lends itself to the expedient computer-aided assessment of the impact of uncertainty on the electromagnetic attributes of the system under study. For example, for the numerical studies presented in this paper, compared to a standard Monte Carlo approach, use of the Smolyak grid and the generated reduced-order model provided for at least two orders of magnitude reduction in the computation of the statistics of the system response.

An important feature of the proposed method is that it is independent of the choice of the algorithm used for model order reduction. Thus, even though for our purposes a Krylov subspace projection scheme was utilized, the proposed methodology can be used in conjunction with all other popular reduced-order modeling algorithms.

The proposed method was used to calculate the impact of statistical variability in the permittivity of the insulating material and the value of the load resistance on the frequency-dependence of the input impedance of a terminated lossless coaxial cable. Through this example it was shown that, when multiple uncertain parameters are involved, prediction of the mean value response of the system using the mean values of the uncertain parameters can be unreliable. Furthermore, it was shown that use of "corner" solutions, utilizing extreme values of the uncertain variables, may lead to gross inaccuracy in the system's response. The second study involved a circular wire loop antenna and it demonstrated that the stochastic reduced-order model can be used to assess the impact of uncertainty of input parameters on the transient response of the antenna.

As already indicated in the introduction, stochastic reduced-order modeling offers a computationally expedient means for system-level electromagnetic analysis in the presence of statistical variability in portions of the overall system. This is a topic of current investigation and will be reported in a forthcoming paper.

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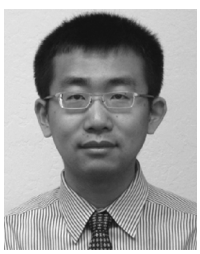
Prasad S. Sumant received dual B.Tech. and M.Tech. degrees from the Indian Institute of Technology Bombay (IITB), Mumbai, India, in 2004, and both the M.S. degree in electrical and computer engineering and the Ph.D. degree in mechanical engineering from the University of Illinois at Urbana-Champaign, in 2010.

He is currently working in the Computational Sciences Function at ExxonMobil Upstream Research Company, Houston, TX. His research interests include scientific computing, computational electromagnetics and numerical methods in geosciences and engineering.

Dr. Sumant was elected to the honor society of Phi Kappa Phi in 2007. He is a recipient of the Outstanding Scholar Fellowship from the Department of Mechanical Science and Engineering at UIUC.



Hong Wu received the B.S. degree in computational mathematics from Nanjing University, China, in 2002 and the Ph.D. degree in electrical and computer engineering from the University of Illinois at Urbana-Champaign, in 2007.

He is currently working in Extreme DA Corporation, Santa Clara, CA. His research interests include FEMs for electromagnetic modeling of integrated RF and high-speed digital circuits, IC capacitance extraction, timing analysis and multi-corner, multi-mode timing optimization.

Andreas C. Cangellaris (F'00) received his Diploma in Electrical Engineering from the Aristotle University of Thessaloniki, Greece in 1981, and the M.S. and Ph.D. degrees in electrical engineering from the University of California, Berkeley, in 1983 and 1985, respectively.

After a two-year tenure with the Electronics Department, General Motors Research Laboratories, he joined the Electrical and Computer Engineering (ECE) Department, University of Arizona, first as Assistant Professor (1987–1992) and then Associate Professor (1992–1997). He joined the ECE Department, University of Illinois, Urbana-Champaign, as Full Professor in 1997, where he is currently the M.E. Van Valkenburg Professor and Department Head of ECE. His expertise and research interests are in applied and computational electromagnetics and its applications to the advancement of modeling methodologies and CAD tools in support of electrical performance analysis and noise-aware design of integrated electronic systems. His research group has conducted pioneering work in the advancement of modeling methodologies and computer tools for predictive electromagnetic response performance of the signal and the power distribution networks of integrated electronics at the board, package and chip levels. Several of the prototype tools developed by his group have been transferred to the microelectronics industry and have contributed to the development of several EDA signal integrity tools. He has authored and coauthored more than 250 papers in peer-reviewed journals and conference proceedings on the topics of finite methods for electromagnetic field modeling; multi-conductor transmission line modeling; and signal and power integrity of integrated electronic systems.

Prof. Cangellaris is a Fellow of IEEE. In 2000, he received the University of Illinois, ECE Department Outstanding Faculty Teaching Award. In 2005, he received the Alexander von Humboldt Research Award for outstanding contributions to electromagnetic theory. He was the co-founder of the IEEE Topical Meeting on Electrical Performance of Electronic Packaging in 1991. He is currently serving as Editor of the *IEEE Press Series on Electromagnetic Wave Theory*. He is an active member of the IEEE Microwave Theory & Techniques Society and the IEEE Components Packaging and Manufacturing Technology Society, contributing to numerous professional activities, conferences and symposia organized and sponsored by them. During 2010–2012 he served as Distinguished Lecturer for the IEEE Microwave Theory & Techniques Society.

Narayana Aluru (M'95) received the B.E. degree (with honors and distinction) from the Birla Institute of Technology and Science (BITS), Pilani, India, in 1989, the M.S. degree from Rensselaer Polytechnic Institute, Troy, NY, in 1991, and the Ph.D. degree from Stanford University, Stanford, CA, in 1995.

He was a Postdoctoral Associate at the Massachusetts Institute of Technology (MIT), Cambridge, from 1995 to 1997. In 1998, he joined the University of Illinois at Urbana-Champaign (UIUC) as an Assistant Professor. He is currently a Professor in the Department of Mechanical Science and Engineering, University of Illinois at Urbana-Champaign. He is also affiliated with the Beckman Institute for Advanced Science and Technology, the Department of Electrical and Computer Engineering and the Bioengineering Department at UIUC.

Prof. Aluru received the NSF CAREER award in 1999, the NCSA faculty fellowship in 1999 and 2006, the 2001 CMES Distinguished Young Author Award, the Xerox Award for Faculty Research in 2002, the ASME Gustus L. Larson Memorial Award in 2006, the USACM Gallagher Young Investigator Award in 2007 and was named a Willett Faculty Scholar by the College of Engineering at UIUC for the period 2002–2008. He is a Subject Editor of the *IEEE/ASME Journal of Microelectromechanical Systems*, served as the Associate Editor of IEEE TRANSACTIONS ON CIRCUITS AND SYSTEMS PART II: EXPRESS BRIEFS for 2004–2005, and currently serves on the Editorial Board of a number of other journals.